An LP approach for computing depth of penetration in piecewise smooth multibody dynamics

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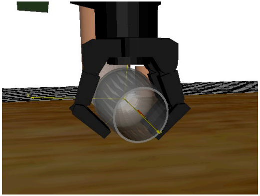
Application of Rigid Multi Body Dynamics

- RMBD in diverse areas
  - ⋆ rock dynamics
  - ⋆ robotic simulations
  - ⋆ virtual reality
  - ⋆ human motion
  - ⋆ nuclear reactors
  - ⋆ haptics
- VR or Virtual reality exposure (VRE) therapy
  - ⋆ fear of heights
  - ⋆ telerehabilitation
  - ⋆ fear of public speaking
  - ⋆ PTSD



What is the model for such problems: DSEC

$$M(q)\frac{d^2q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j=1}^{p} \left(n^{(j)}(q) c_n^{(j)} + D^{(j)}(q)\beta^{(j)} \right) = k(t, q, \frac{dq}{dt})$$

$$\Theta^{(i)}(q) = 0, \qquad i = 1 \dots m$$

$$\Phi^{(j)}(q) \geq 0, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j = 1 \dots p$$

$$\beta = \operatorname{argmin}_{\widehat{\beta}^{(j)}} v^T D(q)^{(j)} \widehat{\beta}^{(j)} \quad \textbf{subject to} \quad \left|\left|\widehat{\beta}^{(j)}\right|\right|_1 \leq \mu^{(j)} c_n^{(j)}, \; j = 1 \dots p$$

$M(q)$: the PD mass matrix, $k(t, q, v)$: external force, $\Theta^{(i)}(q)$: joint constraints.

- Weak solutions can be obtained with time-stepping: which avoids possible lack of strong solutions (Painleve).
- In addition, time-stepping needs one less derivative compared to piecewise DAE stop-restart approaches.
- But this assumes that the gap functions $\Phi^{(j)}$ are easy to compute ... is that the case?

Contact Model

- If we can compute penetration depth d, then nonpenetration constraint is defined by $d = \Phi(q) \geq 0$. Plus, for time-stepping schemes we need derivatives of the penetration depth.
- If the bodies are a sphere of radius R with center at x_S, y_S, z_S and the $z = 0$ hyperplane then the $d = z_S - R$.
- For two spheres of radius R $d = \sqrt{(x_{S_1} - x_{S_2})^2 + (y_{S_1} - y_{S_2})^2 + (z_{S_1} - z_{S_2})^2} - 2R$. It is not always differentiable, but may be for small values of penetration.
- But for most other bodies, it is an extremely painful calculation. And how about the case of convex polyhedra, by far the most widely encountered in apps?

Need to Define and Compute Depth of Penetration

- To avoid infinitely small time steps, say from collisions, then minimum stepsize must exist
- For methods with minimum time step, interpenetration may be unavoidable, thus it needs to be quantified (to limit amount of interpenetration)
- Minimum Euclidean distance good for distance between objects, but not for penetration
- We propose an LP-based approach to compute the penetration depth. We also indicate how to compute "derivatives" which are needed for setting up the time-stepping scheme. Later we compare its theoretical properties with the PD using Minkowski sums

Polyhedra and Expansion/Contraction Maps

Definition

We define CP(A, b, x_o) to be the convex polyhedron P defined by the linear inequalities $Ax \leq b$ with an interior point x_o. We will often just write P = CP(A, b, x_o).

Definition

Let P = CP(A, b, x_o). Then for any nonnegative real number t, the expansion (contraction) of P with respect to the point x_o is defined to be

$$P(x_o, t) = \{x | Ax \leq tb + (1 - t)Ax_o\}$$

So we contract the body until it coincides with x_o, or we extend it to infinity.

Minkowski Penetration Depth

Definition

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2. The Minkowski Penetration Depth (MPD) between the two bodies P_1 and P_2 is defined formally as

$$PD(P_1, P_2) = \min\{||d|| \, | interior(P_1 + d) \bigcap P_2 = \emptyset\}. \quad (1)$$

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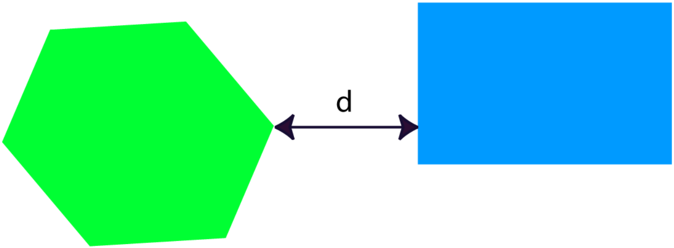


Ratio Metric Penetration Depth

Definition

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2. Then the Ratio Metric between the two sets is given by the LP

$$r(P_1, P_2) = \min\{t | P_1(x_1, t) \bigcap P_2(x_2, t) \neq \emptyset\}, \tag{2}$$

and the corresponding Ratio Metric Penetration Depth (RPD) is given by

$$\rho(P_1, P_2) = \frac{r(P_1, P_2) - 1}{r(P_1, P_2)}. \tag{3}$$

Expansion/Contraction Again

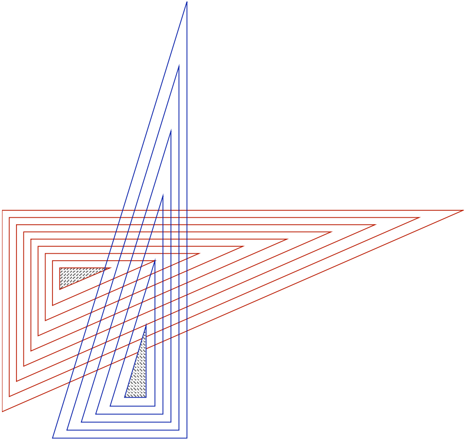


Figure: Visual representation of double expansion or contraction

Metric Equivalence Theorem

Theorem (Metric Equivalence)

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2, s be the MPD between the two bodies, D be the distance between x_1 and x_2, ϵ be the maximum allowable Minkowski penetration between any two bodies. Then the ratio metric penetration depth between the two sets satisfies the relationship

$$\frac{s}{D} \leq \rho(P_1, P_2) \leq \frac{s}{\epsilon}, \quad (4)$$

if P_1 and P_2 have disjoint interiors, and

$$-\frac{s}{\epsilon} \leq \rho(P_1, P_2) \leq -\frac{s}{D} \quad (5)$$

if the interiors of P_1 and P_2 are not disjoint.

Significance of the Metric Equivalence Theorem

- Let number of facets of two polyhedra be m_1 and m_2
  - Computing PD by using the Minkowski sums: $O(m_1^2 + m_2^2)$
  - Fast approximation to PD with stochastic method: $O(m_1^{3/4+\epsilon} m_2^{3/4+\epsilon})$ for any $\epsilon > 0$
  - Solving linear programming problem: $O(m_1 + m_2)$
- $\therefore$ our metric provide us with a simple way to detect collision and measure penetration of two convex polyhedral bodies bodies with lower complexity and is equivalent, for small penetration, to the classical measure
- $\therefore$ for time step h, if the MPD is $O(h^2)$ then so is the RPD
- If we were to use a penalty method with explicity time steps (which is the most common approach, but slow), our job would be done! For everything else we need derivatives!

Nondifferentiability

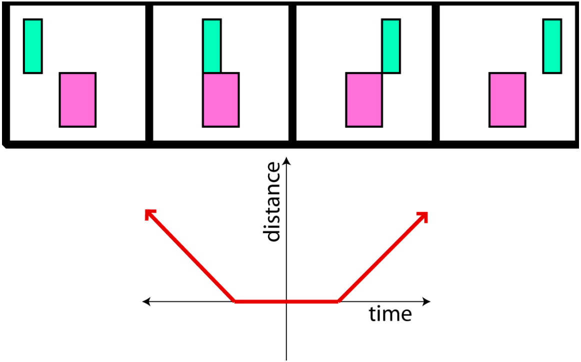


Figure: Nondifferentiability of Euclidean distance function

- Therefore even the Euclidean distance is not differentiable.
- Consider piecewise smooth distance function

Basic Contact Unit

Basic solutions (“basic contact units”, BCU) have a geometrical interpretation: n+1 active constraints, at least one from each polyhedron.

- In 2D: CoF (1,2) In 3D: CoF(1,3), (nonparallel) EoE (2,2)

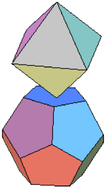


Figure: Corner-on-Face

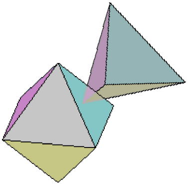


Figure: Edge-on-Edge

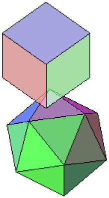


Figure: Face-on-Face

Component Functions

- Associate m^{th} BCU $E^{(m)}$ with component function $\widehat{\Phi}^{(m)}$
- We use the restrictions $P_{E^{(m)}}(x_1, t)$ and $P_{E^{(m)}}(x_2, t)$
- $\widehat{\Phi}^{(m)} = f(r_m)$, where $f(t) = (t-1)/t$ and

$$r_m = \min_{t \geq 0} \left\{ \begin{array}{l} \hat{A}_{m_1} R_1^T x - b_{m_1} t \leq \hat{A}_{m_1} R_1^T x_1 \\ \hat{A}_{m_2} R_2^T x - b_{m_2} t \leq \hat{A}_{m_2} R_2^T x_2 \end{array} \right. \quad (6)$$

and sum of numbers of rows of $\hat{A}_{m_1}$ and $\hat{A}_{m_2}$ is n+1.

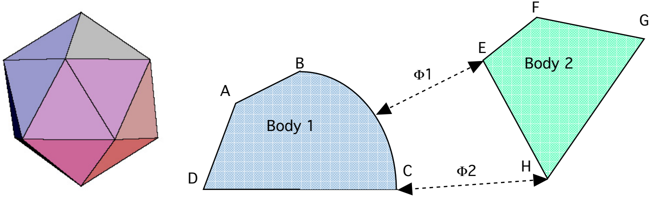


Figure: Uniqueness and Two Component Signed Distance Functions

Max of Component Functions
RPD is the maximum of component distance functions.

Theorem

Suppose $x_1 \neq x_2$ and let $P_i = CP(A_{L_i}R_i^T, b_{L_i} + A_{L_i}R_i^T x_i, x_i)$ be convex polyhedra for $i = 1, 2$ and let $\left\{E^{(1)}, E^{(2)}, \cdots, E^{(N)}\right\}$ be the list of all possible BCUs with corresponding component distance functions $\left\{\widehat{\Phi}^{(1)}, \widehat{\Phi}^{(2)}, \cdots, \widehat{\Phi}^{(N)}\right\}$. Then

$$\rho(P_1, P_2) = \max\left\{\widehat{\Phi}^{(1)}, \widehat{\Phi}^{(2)}, \cdots, \widehat{\Phi}^{(N)}\right\},$$

where $\rho(P_1, P_2)$ is defined by (3).

Differentiability of the Solution of a BCU

$$r(P_E(x_1,t),P_E(x_2,t)) \quad = \quad \min_{t\geq 0}\left\{ \begin{array}{l} \hat{A}_{L_1}R_1^T x - \hat{b}_1 t \leq \hat{A}_{L_1}R_1^T x_1 \\ \hat{A}_{L_2}R_2^T x - \hat{b}_2 t \leq \hat{A}_{L_2}R_2^T x_2 \end{array} \right. \tag{7}$$

Theorem

For any nondegenerate BCU (any COF or nonparallel EoE) with no common face) t is infinitely differentiable, $r(P_E(x_1,t),P_E(x_2,t))$ is infinitely differentiable with respect to the translation vectors and rotation angles.

Generalized Gradient

Lemma

$\Phi^{(j)}$ for $1 \leq j \leq n_B$ is everywhere directionally differentiable. Moreover, the generalized gradient of $\Phi^{(j)}$ is contained in the convex cover of the gradients of its component functions except degenerate ones which are active at q evaluated at q.

Note: We use $\Phi^{(j)o}(q; v) = \limsup_{p \to q, t \downarrow 0} \frac{\Phi^{(j)}(p + tv) - \Phi^{(j)}(p)}{t}$

Noninterpenetration Constraints

- When the penetration depth is differentiable (only one component active), we replace $\Phi^{(j)}(q^{(l+1)} \geq 0$ by $\gamma\Phi^{(j)}(q^{(l)}) + h\nabla_q\Phi^{(j)}(q^{(l)})v \geq 0$. $(0 < \gamma \leq 1)$
- When the penetration depth has multiple components, we replace $\Phi^{(j)}(q^{(l)}) \geq 0$ by $\gamma\Phi^{(j)}(q^{(l)}) + h\nabla_q\widehat{\Phi}^{(j)(m)}(q^{(l)}) \geq 0$, for all active BCU (m) at contact (j), except for the degenerate EoE. It is equivalent to enforcing the inequality for every element of the generalized gradient .
- To allow for relatively large time steps we need to also include the effects of the “almost active constraints” over the generalized gradient.

Active BCUs $\mathcal{E}$
Include set of imminently active BCUs in dynamical resolution.
Determine Set $\mathcal{E}$ by choosing parameters $\hat{\epsilon}_t$ and $\hat{\epsilon}_x$:

$$
\begin{aligned}
\mathcal{E}_1(q) &= \left\{ m \mid \Phi^{(j)} \le \hat{\epsilon}_t,\ j = Bod(E^{(m)}) \right\} \\
\mathcal{E}_2(q) &= \left\{ m \mid 0 \le \widehat{\Phi}^{(m)} - \Phi^{(j)} \le \hat{\epsilon}_t,\ j = Bod(E^{(m)}) \right\} \\
\mathcal{E}_3(q) &= \left\{ m \mid E_x^{(m)} \in CP(A_{L_{m_1}} R_{m_2}^T, b_{L_{m_1}} + A_{L_{m_1}} R_{m_1}^T x_{m_1}, x_{m_1}) + \hat{\epsilon}_x \right\} \\
\mathcal{E}_4(q) &= \left\{ m \mid E_x^{(m)} \in CP(A_{L_{m_2}} R_{m_2}^T, b_{L_{m_2}} + A_{L_{m_2}} R_{m_2}^T x_{m_2}, x_{m_2}) + \hat{\epsilon}_x \right\} \\
\mathcal{E}(q) &= \mathcal{E}_1(q) \bigcap \mathcal{E}_2(q) \bigcap \mathcal{E}_3(q) \bigcap \mathcal{E}_4(q)
\end{aligned}
\tag{8}
$$

$$
\mathcal{A}(q) = \left\{ j \mid \Phi^{(j)}(q) \le \epsilon_t, j = 1, \cdots, p \right\} \tag{9}
$$

Mixed Linear Complementarity Model
Euler discretization of the equations of motion:

$$\begin{aligned} M(q^{(l)})\left(v^{(l+1)} - v^{(l)}\right) &= h_l k\left(t^{(l)}, q^{(l)}, v^{(l)}\right) + \sum_{i=1}^{n_J} c_\nu^{(i)} \nu^{(i)}(q^{(l)}) \\ &+ \sum_{m\in\mathcal{E}} \left(c_n^{(m)} n^{(m)}(q^{(l)}) + \sum_{i=1}^{M_C^{(m)}} \beta_i^{(m)} d_i^{(m)}(q^{(l)}) \right) \end{aligned} \tag{10}$$

Modified linearization of geometrical and noninterpenetration constraints:

$$\begin{aligned} \gamma\Theta^{(i)}(q^{(l)}) + h_l {\nu^{(i)}}^T(q^{(l)}) v^{(l+1)} &= 0, \quad i = 1, 2, \cdots, n_J, \\ {n^{(m)}}^T(q^{(l)}) v^{(l+1)} + \frac{\gamma}{h_l}\Phi^{(j)}(q^{(l)}) &\geq 0 \quad \perp c_n^{(m)} \geq 0, \qquad m \in \mathcal{E}. \end{aligned} \tag{11}$$

Friction Model

Friction model (usual classical pyramid approximation of friction cone, see Stewart & Trinkle 1995 or Anitescu & Hart 2004):

$$\begin{aligned} D^{(m)^T}(q)v + \lambda^{(m)} e^{(m)} &\geq 0 \quad \perp \quad \beta^{(m)} \geq 0, \\ \mu c_n^{(m)} - e^{(m)^T} \beta^{(m)} &\geq 0 \quad \perp \quad \lambda^{(m)} \geq 0. \end{aligned} \tag{12}$$

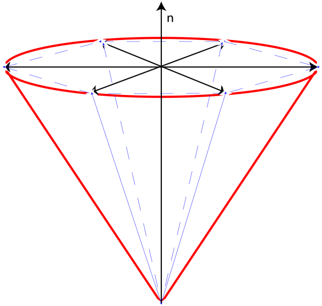


Figure: Approximation of Friction Cone

Definition of Measure of Infeasibility

$$I(q) = \max_{1 \leq j \leq p, 1 \leq i \leq n_J} \left\{ \Phi_{-}^{(j)}(q), \left| \Theta^{(i)}(q) \right| \right\}$$

Assumptions D1 - D3

D1: The mass matrix is constant. That is, $M\left(q^{(l)}\right) = M^{(l)} = M$.

D2: The norm growth parameter is constant: $c(\cdot,\cdot,\cdot) \leq c_o$

D3: The external force is continuous and increases at most linearly with the pos. and vel., and unif. bdd in time:

$$k(t,v,q) = k_o(t,v,q) + f_c(v,q) + k_1(v) + k_2(q)$$

and there is some constant $c_K \geq 0$ such that

$$\begin{array}{rcl} ||k_o(t,v,q)|| & \leq & c_K \\ ||k_1(v)|| & \leq & c_K\,||v|| \\ ||k_2(q)|| & \leq & c_K\,||q||\,. \end{array}$$

Also assume

$$v^T f_c(v,q) = 0 \quad \forall v, q.$$

Algorithm for Piecewise Smooth RMBD

Algorithm

Algorithm for piecewise smooth multibody dynamics

Step 1: Given $q^{(l)}$. $v^{(l)}$. and h_l, calculate the active set $\mathcal{A}\left(q^{(l)}\right)$ and active BCUs $\mathcal{E}\left(q^{(l)}\right)$.

Step 2: Compute $v^{(l+1)}$, the velocity solution of our mixed LCP .

Step 3: Compute $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$.

Step 4: IF finished, THEN stop ELSE set $l = l + 1$ and restart.

Main Result

Theorem

Consider the time-stepping algorithm defined above and applied over a finite time interval [0, T]. Assume that

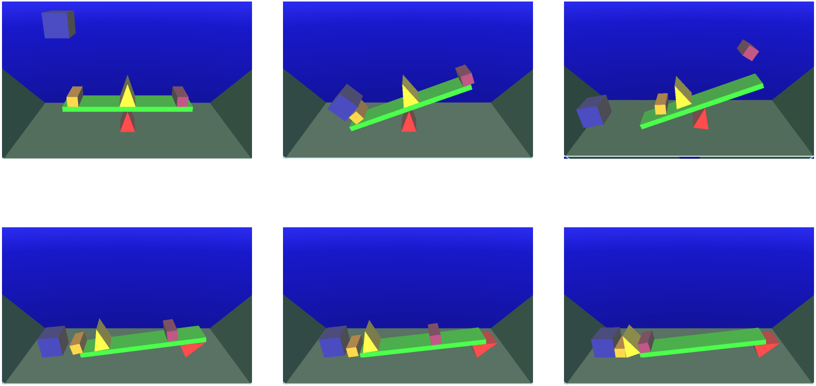
- *The active set $\mathcal{A}(q)$ is defined by (9)*
- *The active BCUs $\mathcal{E}(q)$ are defined by (8)*
- *The time steps $h_l > 0$ satisfy*
$$\sum_{l=0}^{N-1} h_l = T \quad \text{and} \quad \frac{h_{l-1}}{h_l} = c_h, \quad l = 1, 2, \cdots, N-1$$
- *The system satisfies Assumptions (A1) and (D1) - (D3)*
- *The system is initially feasible. That is, $I(q^{(0)}) = 0$*

Then, there exist $H > 0$, $V > 0$, and $C_c > 0$ such that
$$\left|\left|v^{(l)}\right|\right| \leq V \quad \text{and} \quad I\left(q(l)\right) \leq C_c \left|\left|v^{(l)}\right|\right|^2 h_{l-1}^2, \;\; \forall l,\; 1 \leq l \leq N$$

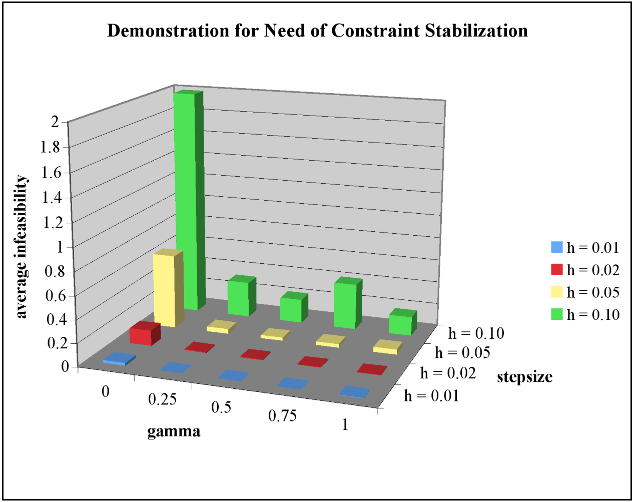
Consequences of the Theorem

- Algorithm achieves constraint stabilization because the infeasibility is bounded above by the size of the solution. In particular, $v^{(l+1)} = 0 \Rightarrow I(q^{(l+1)}) = 0$
- Linear $O(h)$ method yields quadratic $O(h^2)$ infeasibility
- Velocity remains bounded
- No need to change the step size to control infeasibility
- Solve one linear complementarity problem per step

Six successive frames from Balance2



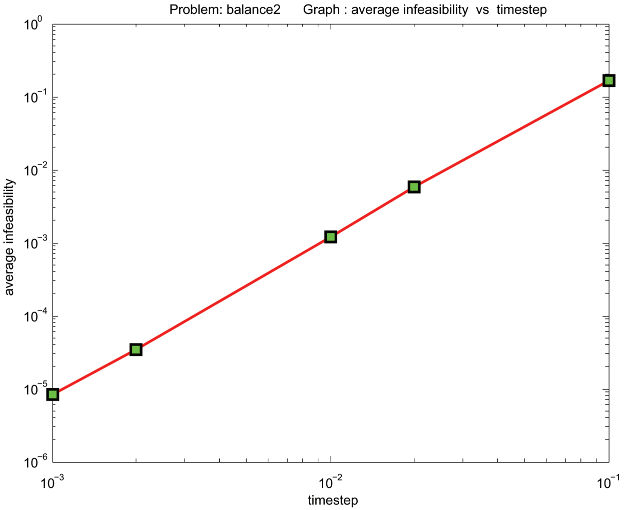
Balance2



Smaller stepsize ⇒ smaller average infeasibility

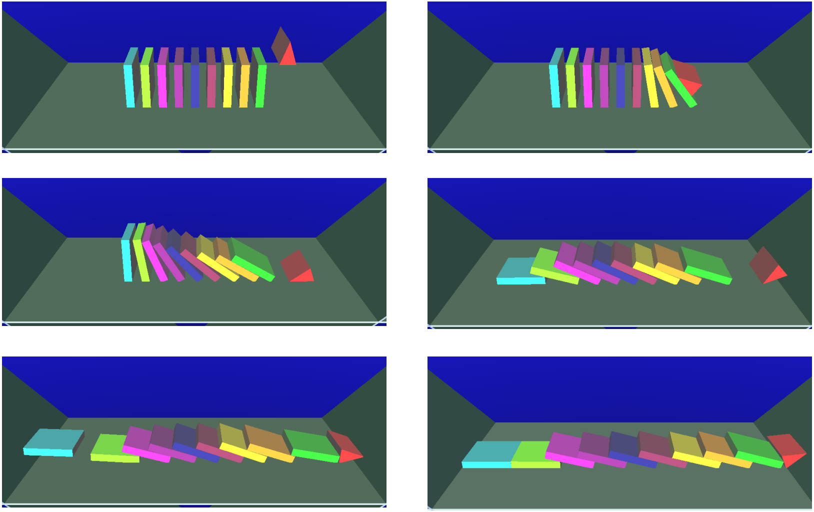
Constraint stabilization ⇒ smaller average infeasibility

Balance2

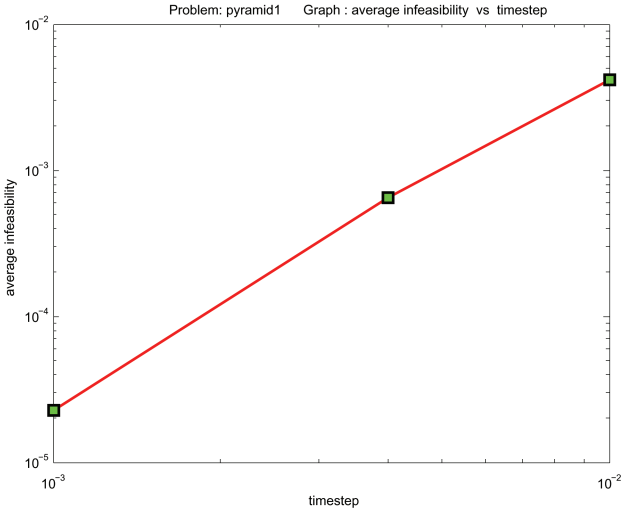


Average infeasibility shows quadratic $O(h^2)$ nature

Six successive frames from Pyramid1

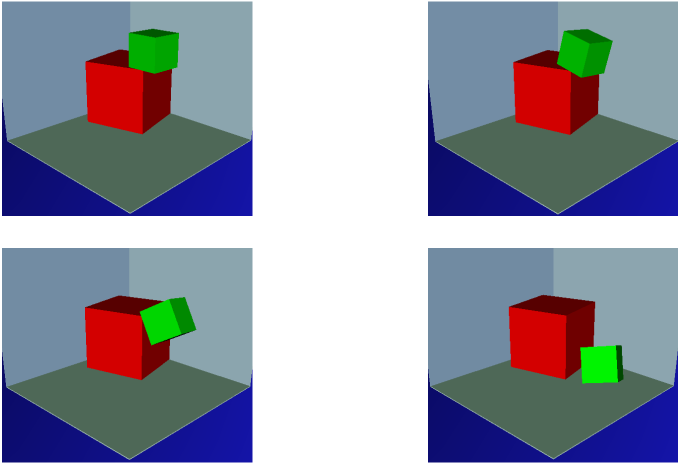


Pyramid1

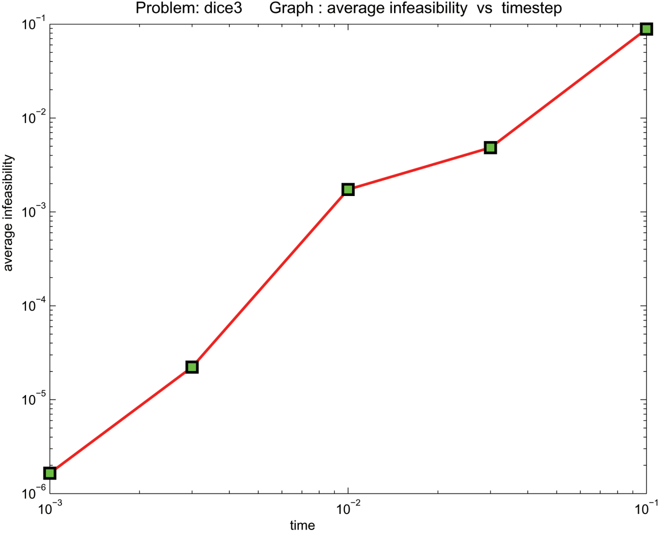


Quadratic convergence of average infeasibility

Four successive frames from Dice3

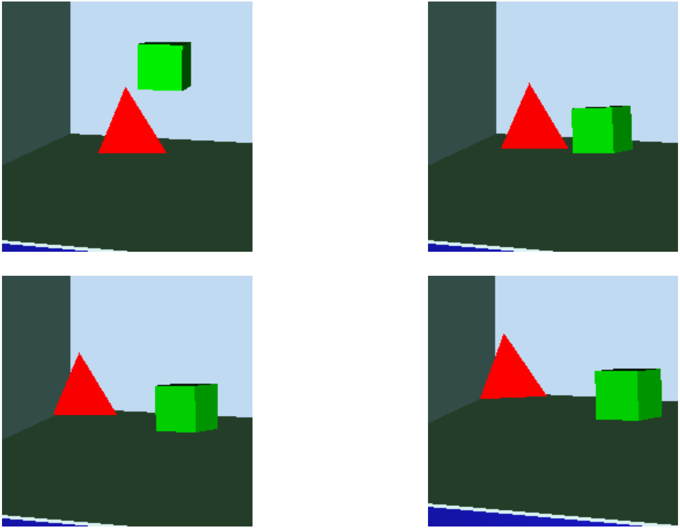


Dice3

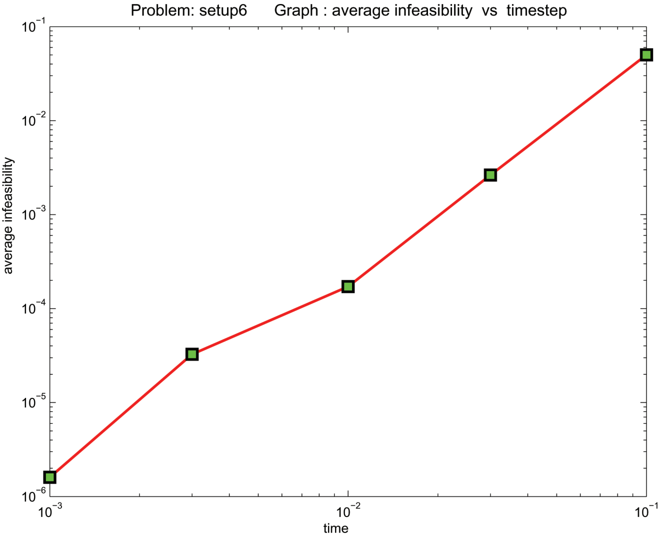


Average infeasibility demonstrates $O(h^2)$ nature

Four successive frames from Setup6



Setup6



Once again, an indication of $O(h^2)$ convergence

Conclusions and Future Research

- We have defined an LP based depth of penetration that is equivalent with Minkowski penetration depth.
- The approach has lower complexity than MPD – linear versus quadratic.
- We have shown how derivative information can be used to achieve constraint stabilization.
- Further research is needed to see if it can also be practically made faster.

Mixed Complementarity and QP Formulation

$$
\begin{array}{lllllll}
M^{(l)}v & -\widetilde{n}\widetilde{c}_n & -\widetilde{D}\widetilde{\beta} & & = -q^{(l)} & & \\
\widetilde{\nu}^T v & & & & = -\Upsilon & & \\
\widetilde{n}^T v & & & -\tilde{\mu}\lambda & \geq -\Gamma - \Delta & \perp & c_n \geq 0 \\
\widetilde{D}^T v & & & +\widetilde{E}\lambda & \geq 0 & \perp & \widetilde{\beta} \geq 0 \\
 & \tilde{\mu}c_n & -\widetilde{E}^T\widetilde{\beta} & & \geq 0 & \perp & \lambda \geq 0
\end{array}
\tag{13}
$$

Mixed Complementarity and QP Formulation

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M^{(l)}v & -\widetilde{n}\widetilde{c}_n & -\widetilde{D}\widetilde{\beta} & & = -q^{(l)} & & & \\
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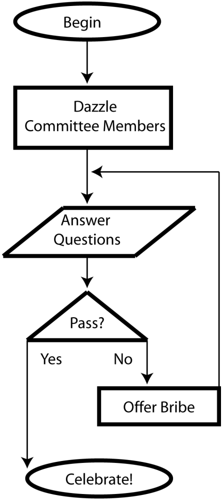
\widetilde{n}^T v & & & -\tilde{\mu}\lambda & \geq -\Gamma - \Delta & \perp & c_n \geq 0 & \\
\widetilde{D}^T v & & & +\widetilde{E}\lambda & \geq 0 & \perp & \widetilde{\beta} \geq 0 & \\
 & \tilde{\mu}c_n & -\widetilde{E}^T\widetilde{\beta} & & \geq 0 & \perp & \lambda \geq 0 &
\end{array} \quad (13)$$

Note (13) constitutes 1^{st}-order optimality conditions of QP

$$\begin{array}{llcll}
\min\limits_{v,\lambda} & \frac{1}{2}v^T M^{(l)} v + q^{(l)^T} v & & & \\
\text{s.t.} & n^{(m)^T} v - \mu^{(m)}\lambda^{(m)} & \geq & -\Gamma^{(m)} - \Delta^{(m)}, & m \in \mathcal{E} \\
 & D^{(m)^T} v + \lambda^{(m)} e^{(m)} & \geq & 0, & m \in \mathcal{E} \\
 & \nu_i^T v & = & -\Upsilon_i, & 1 \leq i \leq n_J \\
 & \lambda^{(m)} & \geq & 0 & m \in \mathcal{E}
\end{array} \quad (14)$$

A constraint-stabilized time-stepping approach for piecewise smooth multibody dynamics

- Ratio Metric
- Differentiability
- Constraints and Model
- Algorithm
- Numerical Results
- Accomplishments



Algorithm for Nearly Active BCUs

Algorithm

Step 1: Solve the dual problem.

Step 2: List the active hyperplanes $H_{1i}, i = 1, \ldots, n_1$ and $H_{2j}, j = 1, \ldots, n_2$.

Step 3: Choose appropriate parameter ϵ,

Step 4a: Check H_{1i} with the list of ϵ adjacent points of H_{2j}.

Step 4b: Check H_{2j} with the list of ϵ adjacent points of H_{1i} .

Step 4c: Check ϵ adjacent edges of H_{1i} and H_{2j}.

Algorithm for Nearly Active BCUs

Algorithm

Step 1: Solve the dual problem.

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Step 4b: Check H_{2j} with the list of ϵ adjacent points of H_{1i} .

Step 4c: Check ϵ adjacent edges of H_{1i} and H_{2j}.

- Because we do not stop nor reduce time steps, we need to include BCUs that would be active at the next step, thus we use “nearly active” BCUs

From of Proof

- Proof proceeds similarly to proof in Anitescu & Hart 2004 and used a Theorem in the same paper
- We use Lebourg's Mean Value Theorem which states that given q_1 and q_2 in the domain of $\Phi^{(j)}$, there exists q_o on the line segment between q_1 and q_2 that satisfies

$$\Phi^{(j)}(q_1) - \Phi^{(j)}(q_2) \in \left\langle \partial\Phi^{(j)}(q_o), q_1 - q_2 \right\rangle .$$

This means that there is some $\Gamma \in \partial\Phi^{(j)}$ such that

$$\Phi^{(j)}(q_1) - \Phi^{(j)}(q_2) = \Gamma(q_1 - q_2).$$

Here $\partial\Phi^{(j)}$ is the generalized gradient.